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O.D.E. of the First order and First degree ①

In this Chapter we will find the solution of the First order, First degree o.D.E. which takes the form:

$$\boxed{y' = f(x,y)} \quad \text{or} \quad \boxed{M(x,y)\,dx + N(x,y)\,dy = 0}$$

* The methods of solution of the first order first degree o.D.E.

① Separable Equations: (فصل المتغيرات)

يكون هذا النوع من المعادلات على الصورة:

حاصل ضرب دالتين

$$\boxed{y' = f(x)\cdot g(y)} \quad \text{or} \quad \boxed{M(x)\,N(y) \ldots = 0}$$

فإذا أمكن وضع المعادلة التفاضلية على هذه الصور

فيكون الحل كما يلي:

$$y' = \frac{dy}{dx} = f(x)\cdot g(y) \Rightarrow \frac{dy}{g(y)} = f(x)\,dx$$

$$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx + C \leftarrow \text{The General Solution}$$

OR when we have the equation:

$$M(x)\,N(y)\,dx + P(x)\,Q(y)\,dy = 0 \Rightarrow M(x)\,N(y)\,dx = -P(x)\,Q(y)\ldots$$

$$\Rightarrow \int \frac{M(x)}{P(x)}\,dx = \int -\frac{Q(y)}{N(y)}\,dy + C$$

EX 1 Solve the DE: $x^2y^2y' + 1 = y$

solution

$\because x^2y^2y' + 1 = y \Rightarrow x^2y^2y' = y - 1$

$\Rightarrow y' = \frac{y-1}{x^2y^2} \Rightarrow \frac{dy}{dx} = \frac{y-1}{x^2y^2}$

$[f(x) \cdot g(y)]$

$\Rightarrow \frac{y^2}{y-1}dy = \frac{dx}{x^2} \Rightarrow \int \frac{y^2}{y-1}dy = \int \frac{dx}{x^2} + C.$

$\Rightarrow \int \left[y + 1 + \frac{1}{y-1}\right]dy = \int x^{-2}dx + C$

$$\begin{array}{r|l} & y+1 \\ y-1 & y^2 \\ & -y^2 - y \\ & y \\ & -y-1 \\ & 1 \end{array}$$

$\Rightarrow \frac{y^2}{2} + y + \ln(y-1) = -\frac{1}{x} + C$ #

(The General solution)

EX 2 Solve the DE: $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

solution

$\int \frac{u' dx}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$

$\because \frac{dy}{dx} = \frac{1+y^2}{1+x^2} \rightarrow f(x) \cdot g(y)$ [separable equation]

$\Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} + C \Rightarrow \boxed{\tan^{-1}y = \tan^{-1}x + C}$ #

EX 3 Solve the initial value problem

$x(1-y^2)y' + (1+x^2)y = 0, \quad y(\frac{1}{2}) = 2$

solution

$x(1-y^2)y' = -(1+x^2)y \Rightarrow \frac{dy}{dx} = -\frac{(1+x^2)y}{x(1-y^2)}$

$\Rightarrow \frac{(1-y^2)dy}{y} = -\frac{(1+x^2)}{x}dx \Rightarrow \int \frac{1-y^2}{y}dy = -\int \frac{1+x^2}{x}dx + C$

$\Rightarrow \int (\frac{1}{y} - y)dy = -\int (\frac{1}{x} + x)dx + C$

$$\ln y - \frac{y^2}{2} = -\left[\ln x + \frac{x^2}{2}\right] + C$$

$\because y(\frac{1}{2}) = 2 \Rightarrow$ at $x = \frac{1}{2} \to y = 2$

$$\Rightarrow \ln 2 - 2 = -\left[\ln \frac{1}{2} + \frac{1}{8}\right] + C$$

$$\Rightarrow C = \ln 2 + \ln \frac{1}{2} - 2 + \frac{1}{8} = \cancel{\ln\left[(2)(\tfrac{1}{2})\right]}^{\,0} - \frac{15}{8} = \frac{-15}{8}$$

$\therefore$ The required particular solution is:

$$\ln y - \frac{y^2}{2} = -\left[\ln x + \frac{x^2}{2}\right] - \frac{15}{8} \quad \#$$

Problems (2) No. (1):

Find the general solution of the given DE:

$$xy\,dx + (x+1)\,dy = 0$$

Solution

لاحظ أن المعادلة المعطاه تأخذ الصورة
$M(x)\,N(y)\,dx + P(x)\,Q(y)\,dy = 0$
∴ فهي separable eqn.

$$\because xy\,dx + (x+1)\,dy = 0 \Rightarrow xy\,dx = -(x+1)\,dy$$

$$\Rightarrow \int \frac{dy}{y} = \int -\frac{x}{x+1}\,dx + C$$

$$\Rightarrow \ln y = -\int \frac{(x+1)-1}{x+1}\,dx + C$$

$$= -\int\left(1 - \frac{1}{x+1}\right)dx + C$$

$$\ln y = -\left[x - \ln(x+1)\right] + C \quad \#$$

Problems ② No. ④ : $2y+(xy+3x)y' = 0$ ④

solution

$$\because y' = \frac{-2y}{xy+3x} = \frac{-2y}{x(y+3)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2y}{x(y+3)} \longrightarrow f(x) \cdot g(y)$$

$$\Rightarrow \frac{(y+3)dy}{y} = \frac{-2}{x}dx \Rightarrow \int \frac{y+3}{y}dy = \int \frac{-2}{x}dx + C$$

$$\Rightarrow \int\left(1+\frac{3}{y}\right)dy = -2\ln x + \ln A$$

$$\Rightarrow y + 3\ln y = \ln(x)^{-2} + \ln A = \ln A x^{-2}$$

$$\Rightarrow \boxed{y + 3\ln y = \ln \frac{A}{x^2}} \quad \#$$

Problems ② No. ⑥ : $y' - xy^2 = 2xy$

solution

$$\because y' = 2xy + xy^2 \Rightarrow \frac{dy}{dx} = x(2y+y^2)$$

$$\Rightarrow \int \frac{dy}{2y+y^2} = \int x\,dx + C$$

$$\Rightarrow \underbrace{\int \frac{dy}{y(2+y)}}_{I} = \frac{x^2}{2} + C \longrightarrow ①$$

$$I: \frac{1}{y(2+y)} = \frac{A}{y} + \frac{B}{2+y}$$ [كسور جزئية]

$* y(2+y)$

$$\Rightarrow 1 = A(2+y) + By$$

let $y=0 \Rightarrow 1 = 2A \Rightarrow \boxed{A = 1/2}$

let $y=-2 \Rightarrow 1 = -2B \Rightarrow \boxed{B = -1/2}$

$$\therefore I = \int \left(\frac{1/2}{y} - \frac{1/2}{2+y} \right) dy = \frac{1}{2}\ln y - \frac{1}{2}\ln(2+y)$$
$$= \ln\sqrt{y} - \ln\sqrt{2+y}$$
$$= \ln\sqrt{\frac{y}{2+y}}$$

in 1 $\Rightarrow$ The General solution is:

$$\ln\left(\sqrt{\frac{y}{2+y}}\right) = \frac{x^2}{2} + C \quad \#$$

Note That: The DE $y' = f(ax+by)$ is reduced to a separable equation by putting $z = ax+by$ (تعويض)

ثم تكمل كما سبق

, if $y' = f(ax+by+c)$ let $z = ax+by+c$

EX 4: Find the general solution of the DE $y' = \cos(x+y)$

Solution

let $z = x+y$

بتفاضل الطرفين بالنسبة الى x

$$\Rightarrow z' = 1 + y' \Rightarrow y' = z' - 1$$

$\therefore$ The DE becomes: $z' - 1 = \cos z$

$$\Rightarrow z' = \cos z + 1 \quad \text{[separable eqn]}$$

$$\Rightarrow \frac{dz}{dx} = \cos z + 1 \Rightarrow \int \frac{dz}{\cos z + 1} = \int dx + C$$

$$\Rightarrow \int \frac{dz}{2\cos^2 \frac{z}{2}} = x + C$$

$$\Rightarrow \frac{1}{2}\int \sec^2 \frac{z}{2}\, dz = x + C$$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$
$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\Rightarrow \tan\frac{z}{2} = x + C \Rightarrow \boxed{\tan\left(\frac{x+y}{2}\right) = x + C} \ \#$$

↑ The general solution

Problem 2 No. 11 : $y' = \sqrt{4x+2y-1}$

solution

(هنا عكن وضع $z = 4x+2y-1$
ولكن للسهولة نضع $z^2 = 4x+2y-1$ للتخلص من الجذر)

$$\text{let } z^2 = 4x+2y-1 .$$

$$\Rightarrow 2zz' = 4 + 2y' \Rightarrow y' = zz' - 2$$

بالتعويض في المعادلة التفاضلية

$$\Rightarrow zz' - 2 = z \Rightarrow \underset{\frac{dz}{dx}}{z'} = \frac{z+2}{z} \quad [\text{separable eqn}$$

$$\Rightarrow \int \frac{z}{z+2}\,dz = \int dx + C$$

$$\Rightarrow \int \frac{(z+2)-2}{z+2}\,dz = x + C.$$

$$\Rightarrow \int \left(1 - \frac{2}{z+2}\right) dz = x + C.$$

$$\Rightarrow z - 2\ln(z+2) = x + C.$$

$$\Rightarrow \sqrt{4x+2y-1} - 2\ln\left[\sqrt{4x+2y-1} + 2\right] = x + C \ \#$$